Maths for Computing
Assignment 3 Solutions

1. *(3 marks)* Prove or disprove the following statement. The set of real numbers containing only a finite number of 1s in their decimal representation is countable.
(Numbers allowed in the sets are 1.11, 11.1, 1.111, etc. Numbers not allowed in the set are 1.11…, 1.2, 3.4111…, etc.)
Solution: The set is countable. We can arrange all the real numbers containing only finitely many 1s in a matrix. In the ith row of this matrix, we have, in increasing order, real numbers that have $i - 1$ many $1s$ before decimal.

$$\begin{array}{l} 0, .1,\ .11,\ .111,\ .1111, \dots \\ 1,\ 1.1,\ 1.11,\ 1.111,\ 1.1111, \dots \\ 11, 11.1,\ 11.11,\ 11.111,\ 11.1111, \dots \\ \vdots \end{array}$$

Any real number that contains i many 1s before decimal and total j many $1s$ after decimal will be present in the $(i + 1)$th row and $(j + 1)$th column. (It's ok, if you have missed 0. No marks will be deducted for that.)

Now we can simply traverse all the numbers in the matrix in a dove-tail order to create a sequence of real numbers that contains only finitely many 1s. Since we have a sequence of such real numbers, their set is countable.

2. *(5 marks)* Let $X = \{x \mid x \text{ is a real number such that } 1 < x < 2\}$. Then prove that $|X| = |R|$.
Solution: *This turned out to be a lot more easier than I predicted.*
We can prove $|X| = |R|$ using Schröder-Bernstein theorem. That is, we will give an injection from X to R and from R to X.

Injection from X to R: $f : X \to R$ is $f(x) = x$. Clearly, f is an injection.

Injection from R to X: $f : R \to X$ is $f(x) = \dfrac{2^x}{1 + 2^x} + 1$.

Range of f is X: Since 2^x is positive for any $x \in R$, $\frac{2^x}{1+2^x} > 0$. Therefore, $\frac{2^x}{1+2^x} + 1 > 1$. Also, $2^x < 1 + 2^x \implies \frac{2^x}{1+2^x} < 1 \implies \frac{2^x}{1+2^x} + 1 < 2$. Hence, $f(x) < 2$.

f is one-to-one: Let $x, y \in R$ such that $f(x) = f(y)$. Then,

$$
\begin{aligned}
& \frac{2^x}{1+2^x} + 1 = \frac{2^y}{1+2^y} + 1 \\
\implies & \frac{2^x}{1+2^x} = \frac{2^y}{1+2^y} \\
\implies & \frac{1+2^x}{2^x} = \frac{1+2^y}{2^y} \\
\implies & 1 + \frac{1}{2^x} = 1 + \frac{1}{2^y} \\
\implies & \frac{1}{2^x} = \frac{1}{2^y} \\
\implies & 2^x = 2^y \\
\implies & x = y \qquad \text{(take log on both sides.)}
\end{aligned}
$$